

Revisiting Canonical Quantization

John R. Klauder*

Department of Physics and
Department of Mathematics
University of Florida
Gainesville, FL 32611-8440

Abstract

Conventional canonical quantization procedures directly link various c -number and q -number quantities. Here, we advocate a different association of classical and quantum quantities that renders classical theory a natural subset of quantum theory with $\hbar > 0$, in conformity with the real world wherein nature has chosen $\hbar > 0$ rather than $\hbar = 0$. While keeping the good results of conventional procedures, some examples are presented for which the new procedures offer better results than conventional ones.

1 INTRODUCTION

The most common approach to a quantum theory is through Schrödinger's equation

$$i\hbar \partial \psi(x, t) / \partial t = \mathcal{H}(-i\hbar \partial / \partial x, x) \psi(x, t) , \quad (1)$$

illustrated for a single degree of freedom. Here the function $\mathcal{H}(p, q)$ generally differs from the classical (c) Hamiltonian $H_c(p, q)$ by terms of order \hbar , the coordinates $p \rightarrow -i\hbar \partial / \partial x$ and $q \rightarrow x$, and $\int_{-\infty}^{\infty} |\psi(x, t)|^2 dx = 1$. A similar prescription applies to classical systems with N degrees of freedom,

*Email: klauder@phys.ufl.edu

$N \leq \infty$. Although this scheme is widely successful, there are certain questionable aspects. Generally, the given procedure works well only for certain canonical coordinate systems, namely, for “Cartesian coordinates” [1], despite the fact that the classical phase space has a symplectic structure (e.g., $dp\,dq$, interpreted as an element of surface area) but *no* metric structure (e.g., $dp^2 + (p^2 + 1)dq^2$) [2]. Moreover, for certain classical systems, and even when using the correct coordinates to provide a canonical quantization, a subsequent classical limit in which $\hbar \rightarrow 0$ leads to a manifestly *different* classical system from the original one, thus violating the eminently natural rule that de-quantization should lead back to the original classical system. Finally, the classical framework for which $\hbar = 0$ is fundamentally different from the quantum framework for which $\hbar > 0$, and it is the latter realm that characterizes the real world.

There are several arguments that support this new approach. First, by analogy, note that the real world is relativistic in character, but a nonrelativistic approximation to classical mechanics can be made within relativistic classical mechanics *without changing the formulation* and keeping the speed of light c fixed and finite. Likewise, since the real world is also governed by quantum mechanics, it is necessary that classical mechanics somehow be contained within quantum mechanics in such a way that *it involves the same formulation* and keeps the reduced Planck’s constant $\hbar = h/2\pi$ fixed and nonzero. Second, the present prescription on how a classical system should be quantized leads—for some problems, e.g., self-interacting scalar fields φ_n^4 , for spacetime dimensions $n \geq 4$ —to unnatural behavior in that the classical limit of a conventional quantization does *not* reduce to the original classical model when $\hbar \rightarrow 0$! As we shall illustrate, this serious discrepancy can be overcome with procedures analogous to those discussed in the present article. However, we do not need to invoke complicated examples to learn some of the advantages of a new way to link classical and quantum systems together.

2 ENHANCED CANONICAL QUANTIZATION

As discussed above, we shall propose a different manner of quantization—called *Enhanced Quantization*—that keeps all the good results of conventional canonical quantization, but offers better solutions when needed. We start

with the quantum action functional A_Q given, for a single degree of freedom and normalized wave functions, by

$$A_Q = \int_0^T \int_{-\infty}^{\infty} \psi^*(x, t) [i\hbar \partial/\partial t - \mathcal{H}(-i\hbar \partial/\partial x, x)] \psi(x, t) dx dt \quad (2)$$

from which Schrödinger's equation (1) may be derived by a general stationary variation, $\delta A_Q = 0$, provided that $\delta\psi(x, 0) = 0 = \delta\psi(x, T)$. Such variations mirror variations that can actually be realized in practice, but, sometimes, not all variations are possible. As an example, consider a *microscopic system*: then while *microscopic observations* can make sufficiently general variations to deduce Schrödinger's equation, *macroscopic observations* are confined to a smaller subset of possible variations. For example, we include only those that can be realized without disturbing the system, such as changes made in accordance with Galilean invariance, namely a change in position by q and a change in momentum by p (as realized by a change in velocity). Choosing a foundation on which to build, we select a basic normalized function—call it $\eta(x)$ —which, when transported by p and q as noted above, gives rise to a family of functions $\eta_{p,q}(x) \equiv e^{ip(x-q)/\hbar} \eta(x-q)$, where $-\infty < p, q < \infty$ [note: $\eta_{0,0}(x) \equiv \eta(x)$]. Within the context of quantum mechanics, these functions are also well known as *canonical coherent states* [3], and the basic function is generally referred to as the fiducial vector, all expressed here in the Schrödinger representation. While not required, it is useful to impose $\int_{-\infty}^{\infty} x |\eta(x)|^2 dx = 0$ as well as $\int_{-\infty}^{\infty} \eta(x)^* \eta'(x) dx = 0$, called “physical centering”, which then leads to

$$\int_{-\infty}^{\infty} x |\eta_{p,q}(x)|^2 dx = q, \quad -i\hbar \int_{-\infty}^{\infty} \eta_{p,q}(x)^* \eta'_{p,q}(x) dx = p, \quad (3)$$

relations that fix the physical meaning of p and q , independently of the basic function $\eta(x)$. Finally, being unable as macroscopic observers to vary the functions $\psi(x, t)$ in (2) arbitrarily, we restrict (R) the set of allowed variational states so that $\psi(x, t) \rightarrow \eta_{p(t),q(t)}(x)$, which leads to

$$\begin{aligned} A_{Q(R)} &= \int_0^T \int_{-\infty}^{\infty} \eta_{p(t),q(t)}(x)^* [i\hbar \partial/\partial t - \mathcal{H}(-i\hbar \partial/\partial x, x)] \eta_{p(t),q(t)}(x) dx dt \\ &= \int_0^T [p(t) \dot{q}(t) - H(p(t), q(t))] dt, \end{aligned} \quad (4)$$

where

$$H(p, q) \equiv \int_{-\infty}^{\infty} \eta(x)^* \mathcal{H}(p - i\hbar \partial/\partial x, q + x) \eta(x) dx. \quad (5)$$

Because $\hbar > 0$, we call (4) and (5) the *enhanced classical action functional*. Assuming, for simplicity, that the Hamiltonian operator is a polynomial, it readily follows that

$$H(p, q) = \mathcal{H}(p, q) + \mathcal{O}(\hbar; p, q) \quad (6)$$

for many choices of the basic function $\eta(x)$. In such cases, the strictly classical action functional arises if the limit $\hbar \rightarrow 0$ is applied to (4) and (5).

The introduction of classical canonical coordinate transformations—for example, $(p, q) \rightarrow (\tilde{p}, \tilde{q})$ such that $p dq = \tilde{p} d\tilde{q} + d\tilde{G}(\tilde{p}, \tilde{q})$, and for which we choose $\tilde{\eta}_{\tilde{p}, \tilde{q}}(x) \equiv \eta_{p(\tilde{p}, \tilde{q}), q(\tilde{p}, \tilde{q})}(x) = \eta_{p, q}(x)$ —leads to an enhanced classical action functional properly expressed in the new coordinates, but with *no* change of the quantum formalism—nor of the physics—whatsoever.

Stationary variation of the enhanced classical action functional leads to Hamilton's equations of motion based on the enhanced classical Hamiltonian $H(p, q)$. In this sense we have shown that a suitably restricted domain of the *quantum action functional*, consisting of a two-dimensional, continuously connected sheet of functions, leads to an enhanced *canonical classical action functional*, with the benefit that $\hbar > 0$ still, and, in this way, we have achieved the goal of *embedding classical mechanics within quantum mechanics*! Moreover, the enhanced classical equations of motion may have $\mathcal{O}(\hbar)$ correction terms—the form of which is dictated by (5)—that may be of interest in modifying the strictly classical solutions in interesting ways. One such example is offered below

In a crude sense, and for a reasonable range of (p, q) , elements of the set of states $\{\eta_{p, q}(x)\}$ act like the illumination from a flashlight used by a burglar in peering through the window of a deserted house on a pitch black night; indeed, the role of physically centering the function $\eta(x)$ is like ensuring the flashlight is aimed through the window and not at the brick walls. Changing the basic function $\eta(x)$ is like changing the orientation or the cone of illumination of the flashlight. Quantum mechanically, whatever the choice of $\eta(x)$, the set of functions $\{\eta_{p, q}(x)\}$, for all (p, q) , span the space of all square integrable functions, $L^2(\mathbb{R})$. In examining the enhanced classical theory, however, some choices of $\eta(x)$ may be better than others.

A common choice for $\eta(x)$ —which acts like a bright, narrow-beam flashlight—is given, with $\omega > 0$, by

$$\eta(x) = (\omega/\pi\hbar)^{1/4} e^{-\omega x^2/2\hbar} \quad (7)$$

for which $H(p, q)$ satisfies (6), i.e., $\mathcal{H}(p, q) = H(p, q)$ up to terms of order \hbar . This last property is exactly what is meant by having “Cartesian coordinates”, although such coordinates can not originate from the classical phase space. However, Cartesian coordinates do have a natural origin from an enhanced quantization viewpoint. The set of allowed variational states $\{\eta_{p,q}(x)\}$ —a set of canonical coherent states as noted earlier—forms a two-dimensional, continuously connected sheet of normalized functions within the set of normalized square integrable functions, and a natural metric can be given for such functions. Since the overall phase of any wave function carries no physics, the (suitably scaled) square of the distance between two coherent-state rays is given by

$$d_R^2(p', q'; p, q) \equiv (2\hbar) \min_{\alpha} \int_{-\infty}^{\infty} |\eta_{p',q'}(x) - e^{i\alpha} \eta_{p,q}(x)|^2 dx, \quad (8)$$

which for two infinitesimally close coherent-state rays becomes (this is also the Fubini-Study metric [4])

$$d\sigma^2(p, q) \equiv (2\hbar) [\int |d\eta_{p,q}(x)|^2 dx - |\int \eta_{p,q}(x)^* d\eta_{p,q}(x) dx|^2], \quad (9)$$

and for $\eta(x) = (\omega/\pi\hbar)^{1/4} \exp(-\omega x^2/2\hbar)$, (9) reduces to

$$d\sigma^2(p, q) = \omega^{-1} dp^2 + \omega dq^2, \quad (10)$$

which ensures that p and q are Cartesian coordinates. Although this metric originates in the quantum theory with the canonical coherent states, it may also be assigned to the classical phase space as well.

At this point we have recreated conventional canonical quantization in that we have identified canonical variables p and q that behave as Cartesian coordinates, and for which the quantum Hamiltonian is effectively the same as the conventionally chosen one—particularly for classical Hamiltonians of the form $p^2/(2m) + V(q)$ —especially if ω in (7) is chosen very large. Thus, enhanced quantization can reproduce conventional canonical quantization—*but it has other positive features as well!*

3 ENHANCED AFFINE QUANTIZATION

Classical canonical variables p and q fulfill the Poisson bracket $\{q, p\} = 1$, which translates to the Heisenberg commutation rule $[x, -i\hbar(\partial/\partial x)] = i\hbar$;

these operators generate the transformations that characterize the canonical coherent states. Multiplying the Poisson bracket by q leads to $\{q, pq\} = q$, which corresponds to $-(i\hbar/2)[x, x(\partial/\partial x) + (\partial/\partial x)x] = i\hbar x$ after both sides of the commutator are multiplied by x . This expression, *derived from the Heisenberg commutation relation*, is known as an affine commutation relation between affine variables. While the operator $-i\hbar(\partial/\partial x)$ acts to generate *translations* of x , the operator $-(i\hbar/2)[x(\partial/\partial x) + (\partial/\partial x)x]$ acts to generate *dilations* of x . If one deals with a classical variable $q > 0$ and its quantum analog $x > 0$ (both chosen dimensionless for convenience), then the canonical coherent states are unsuitable and we need a different set of coherent states. We choose a new basic function $\xi(x) \equiv M x^{\tilde{\beta}/\hbar - 1/2} e^{-\tilde{\beta}x/\hbar}$, $\tilde{\beta} > 0$ and $x > 0$, with M a normalization factor, for which it follows that $\int_0^\infty x |\xi(x)|^2 dx = 1$ and $\int_0^\infty \xi(x)^* [x(\partial/\partial x) + (\partial/\partial x)x] \xi(x) dx = 0$. We also introduce suitable *affine coherent states* as $\xi_{p,q}(x) \equiv q^{-1/2} e^{ip(x-q)/\hbar} \xi(x/q)$, where $q > 0$ and $-\infty < p < \infty$ [note: $\xi_{0,1}(x) \equiv \xi(x)$]. The affine coherent states involve translation in Fourier space by p and dilation—i.e., (de)magnification, partially realized by a magnifying glass—in configuration space by q , and they describe a new, two-dimensional, continuously connected sheet of normalized functions.

The quantum action functional on the half space $x > 0$ is given by

$$A'_Q = \int_0^T \int_0^\infty \phi(x, t)^* [i\hbar \partial/\partial t - \mathcal{H}(-i\hbar \partial/\partial x, x)] \phi(x, t) dx dt, \quad (11)$$

and a suitable stationary variation leads to Schrödinger's equation. Restricted (R) to the affine coherent states, we find that

$$\begin{aligned} A'_{Q(R)} &= \int_0^T \int_0^\infty \xi_{p(t), q(t)}(x)^* [i\hbar \partial/\partial t - \mathcal{H}(-i\hbar \partial/\partial x, x)] \xi_{p(t), q(t)}(x) dx dt \\ &= \int_0^T [p(t) \dot{q}(t) - H(p(t), q(t))] dt, \end{aligned} \quad (12)$$

where

$$H(p, q) \equiv \int_0^\infty \xi(x)^* \mathcal{H}(p - iq^{-1}\hbar \partial/\partial x, qx) \xi(x) dx. \quad (13)$$

Equations (12) and (13) strongly suggest that they correspond to an enhanced *canonical classical action functional*. In other words, enhanced quantization has found a different, two-dimensional sheet of normalized functions that nevertheless has a canonical system as the classical limit. Invariance under canonical coordinate transformations follows along the same lines as before. For this system, “Cartesian coordinates” are *not* appropriate; instead

the geometry of the affine coherent-state rays leads to a Fubini-Study metric given by

$$\begin{aligned} d\sigma^2(p, q) &\equiv (2\hbar) [\int |d\xi_{p,q}(x)|^2 dx - | \int \xi_{p,q}(x)^* d\xi_{p,q}(x) dx |^2] \\ &= \tilde{\beta}^{-1} q^2 dp^2 + \tilde{\beta} q^{-2} dq^2 , \end{aligned} \quad (14)$$

which is a space of constant negative curvature: $-2/\tilde{\beta}$ (a Poincaré half plane). As with the canonical case, this new geometry can be added to the classical phase space if so desired.

While we have focussed on the case where $x > 0$, it is also possible to consider $x < 0$, and even a *reducible representation* case where $|x| > 0$, in which case, affine quantization can, effectively, replace canonical quantization.

4 EXAMPLES

4.1 Model one

Consider the classical action functional for a single degree of freedom given by

$$A_C = \int_0^T [p\dot{q} - qp^2] dt , \quad (15)$$

with the physical requirement that $q > 0$. The classical solutions for this example are given by

$$p(t) = p_0(1 + p_0 t)^{-1} , \quad q(t) = q_0(1 + p_0 t)^2 , \quad (16)$$

where (p_0, q_0) denote initial data at $t = 0$. Although $q(t)$ is never negative, *every solution* with positive energy $E = q_0 p_0^2$ becomes singular since $q(-p_0^{-1}) = 0$.

We like to believe that quantization of singular classical systems may, sometimes, eliminate the singular behavior, and let us see if that can occur for the present system. Conventional canonical quantization is ambiguous up to terms of order \hbar , and that makes it difficult to decide on proper semi-classical corrections with $\hbar > 0$. In contrast, enhanced quantization always keeps $\hbar > 0$ and points to quite specific semi-classical corrections. Adopting enhanced affine quantization, the enhanced classical action functional becomes

$$\begin{aligned} A_{Q(R)} &= \int_0^T \int_0^\infty \xi_{p(t),q(t)}(x)^* [i\hbar\partial/\partial t - (-i\hbar\partial/\partial x)x(-i\hbar\partial/\partial x)] \xi_{p(t),q(t)}(x) dx \\ &= \int_0^T [p(t)\dot{q}(t) - q(t)p(t)^2 - Cq(t)^{-1}] dt , \end{aligned} \quad (17)$$

where $C \equiv \hbar^2 \int_0^\infty x |\xi'(x)|^2 dx > 0$. With our convention that q and x are dimensionless, it follows that the dimensions of C are those of \hbar^2 . While the numerical value of C may depend on $\xi(x)$, the modification of the classical Hamiltonian has just one term proportional to $\hbar^2 q^{-1}$, which guarantees that the semi-classical solutions do indeed eliminate the divergences encountered in the strictly classical solutions.

Model one is based on [5], which also includes additional details.

4.2 Model two

This example involves another feature of enhanced quantization besides that of affine quantization. Although somewhat more advanced than Model one, the present example illustrates a significant advantage of enhanced quantization, so we choose to include it. Model two also involves many (possibly, infinitely many) degrees of freedom: $N \leq \infty$. When $N = \infty$, it also provides a valid quantization of a system that fails to be properly quantized by conventional quantization techniques. The model in question has a classical Hamiltonian given by

$$H(\vec{p}, \vec{q}) = \frac{1}{2}[\vec{p}^2 + m_0^2 \vec{q}^2] + \lambda_0 (\vec{q}^2)^2, \quad (18)$$

where $\vec{p} = \{p_1, p_2, \dots, p_N\}$, $\vec{p}^2 \equiv \vec{p} \cdot \vec{p} = \sum_{n=1}^N p_n^2$ (and likewise for \vec{q}); when $N = \infty$, it is understood that $\vec{p}^2 + \vec{q}^2 < \infty$. Canonical quantization of this model for $N = \infty$ leads to a *free* quantum theory with no real quartic interaction, which inevitably passes to a classical free model as $\hbar \rightarrow 0$, and thus is different from the original nonfree classical model. This unnatural behavior happens because, as N becomes large, nascent divergences caused by the interaction term must be tamed by rescaling $\lambda_0 \rightarrow \lambda_0/N$, eventually nullifying any real influence of the interaction.

We may arrive at the same conclusion from a different line of reasoning. The difficulties for this model arise when $N = \infty$, but they can be analyzed and dealt with when $1 \ll N < \infty$. We assume that the ground state $\langle \vec{x} | 0 \rangle = \psi_0(\vec{x})$ for this problem is unique and has the full rotational symmetry of the model. Thus the characteristic function (i.e., the Fourier transform) of the ground-state distribution becomes

$$\begin{aligned} C(\vec{p}) &= \int e^{i \vec{p} \cdot \vec{x} / \hbar} \psi_0(\vec{x})^2 d^N x \\ &= \int e^{i p_r r \cos(\theta) / \hbar} \rho(r) r^{N-1} \sin(\theta)^{N-2} dr d\theta d\Omega_{N-2} \end{aligned}$$

$$\simeq \int e^{-p_r^2 r^2 / \hbar^2} w(r) dr \quad (19)$$

where $p_r \equiv \sqrt{\vec{p}^2}$, $r \equiv \sqrt{\vec{x}^2}$, $d\Omega_{N-2}$ refers to the other angular variables, and the final approximation is based on a steepest descent evaluation of the θ integral for large N . Thus, in the limit $N \rightarrow \infty$ the most general form of the characteristic function is given by

$$C(\vec{p}) = \int_0^\infty e^{-b \vec{p}^2 / \hbar} d\mu(b) \quad (20)$$

for some normalized, positive measure μ . While this analysis deals with preserving full symmetry, the general result in (20) does not respect uniqueness of the desired ground state (or equivalently, irreducible position operators). To obtain uniqueness of the ground state, the measure $\mu(b)$ must fulfill $d\mu(b) = \delta(b - 1/4m') db$, which leads to $C(\vec{p}) = \exp[-\vec{p}^2/4m'\hbar]$, and corresponds to a free system for some positive mass m' .

In our discussion of this model below, we mostly use an abstract quantum notation for brevity. We define coherent states as

$$|\vec{p}, \vec{q}\rangle \equiv \exp[-i\vec{q} \cdot \vec{P}/\hbar] \exp[i\vec{p} \cdot \vec{Q}/\hbar] |0\rangle, \quad (21)$$

where $(m\vec{Q} + i\vec{P})|0\rangle = 0$, and it follows that

$$\begin{aligned} & \langle \vec{p}', \vec{q}' | \vec{p}, \vec{q} \rangle \\ &= \exp\{i(p' + p) \cdot (q' - q)/2\hbar - [(1/m)(p' - p)^2 + m(q' - q)^2]/4\hbar\}, \end{aligned} \quad (22)$$

an expression ensuring irreducible canonical operators \vec{Q} and \vec{P} , for which $[Q_m, P_n] = i\hbar\delta_{m,n}\mathbb{1}$, $1 \leq m, n \leq N$. However, for this model, irreducible operators only apply to free models, and so we will need to consider suitable *reducible* position and momentum operator representations, for which, effectively, $|0\rangle$ is replaced by $|0; (K)\rangle$, such that

$$\begin{aligned} & \langle \vec{p}', \vec{q}'; (K) | \vec{p}, \vec{q}; (K) \rangle \\ &= \exp\{i(p' + p) \cdot (q' - q)/2\hbar - [(K/m)(p' - p)^2 + m(q' - q)^2]/4\hbar\}, \end{aligned} \quad (23)$$

where $K > 1$.

A detailed derivation of a proper quantization of this model is presented elsewhere; see [6, 7]. Here, our main goal is to present the solution and show that it conforms to enhanced quantization and not to conventional quantization. To construct suitable reducible operator representations, we

introduce another independent set of classical and quantum variables, \vec{r}, \vec{s} and \vec{R}, \vec{S} , with $[S_m, R_n] = i\hbar\delta_{m,n}\mathbb{1}$, $1 \leq m, n \leq N$. For $N < \infty$, we choose a basic vector $|0; \zeta\rangle$ given in the Schrödinger representation by

$$\langle \vec{x}, \vec{y} | 0; \zeta \rangle \equiv \psi_{0;\zeta}(\vec{x}, \vec{y}) \equiv M' \exp[-m(\vec{x}^2 + 2\zeta \vec{x} \cdot \vec{y} + \vec{y}^2)/2\hbar], \quad (24)$$

where $0 < \zeta < 1$; the case $\zeta = 0$ leads to irreducible operators. There are two “free-looking” Hamiltonian operators with this vector as their common, unique ground state, namely $\mathcal{H}_p \equiv \frac{1}{2} : [\vec{P}^2 + m(\vec{Q} + \zeta \vec{S})^2] :$ and $\mathcal{H}_r \equiv \frac{1}{2} : [\vec{R}^2 + m(\vec{S} + \zeta \vec{Q})^2] :$, where $:(\cdot):$ signifies normal ordering. We choose new coherent states based on $\psi_{0;\zeta}$ given by

$$|\vec{p}, \vec{q}; \zeta\rangle \equiv \exp[-i\vec{q} \cdot \vec{P}/\hbar] \exp[i\vec{p} \cdot \vec{Q}/\hbar] |0; \zeta\rangle, \quad (25)$$

and we consider the operator

$$\mathcal{H}_1 \equiv \mathcal{H}_p + \mathcal{H}_r + 4\nu : \mathcal{H}_r^2 :. \quad (26)$$

It follows directly that

$$\begin{aligned} \langle \vec{p}', \vec{q}'; \zeta | \mathcal{H}_1 | \vec{p}, \vec{q}; \zeta \rangle &= \{ \frac{1}{2} [(m\vec{q}' - i\vec{p}) \cdot (m\vec{q} + i\vec{p}) + m^2\zeta^2 \vec{q}' \cdot \vec{q}] \\ &\quad + \nu\zeta^4 m^4 (\vec{q}' \cdot \vec{q})^2 \} \langle \vec{p}', \vec{q}'; \zeta | \vec{p}, \vec{q}; \zeta \rangle, \end{aligned} \quad (27)$$

where

$$\begin{aligned} \langle \vec{p}', \vec{q}'; \zeta | \vec{p}, \vec{q}; \zeta \rangle &\equiv \exp\{i(\vec{p}' + \vec{p}) \cdot (\vec{q}' - \vec{q})/2\hbar \\ &\quad - [(\vec{p}' - \vec{p})^2/4m(1 - \zeta^2) + m(\vec{q}' - \vec{q})^2/4\hbar]\} \\ &= \exp[i(\vec{p}' + \vec{p}) \cdot (\vec{q}' - \vec{q})/2\hbar] \\ &\times M''(\zeta) \int \exp\{i(\vec{p}' - \vec{p}) \cdot \vec{y}/\hbar - [(\vec{p}' - \vec{p})^2/4m\hbar \\ &\quad + m\hbar(\vec{q}' - \vec{q})^2/4\hbar + m(\zeta^{-2} - 1)\vec{y}^2/\hbar]\} d^N y. \end{aligned} \quad (28)$$

In comparison with (23), it follows that $K = (1 - \zeta^2)^{-1} > 1$, and the integral representation in the last line of this equation illustrates a superposition over irreducible representations leading to the desired reducible representation.

Finally, we observe that

$$\begin{aligned} \langle \vec{p}, \vec{q}; \zeta | \mathcal{H}_1 | \vec{p}, \vec{q}; \zeta \rangle &= \frac{1}{2} [\vec{p}^2 + (1 + \zeta^2)m^2 \vec{q}^2] + \nu\zeta^4 m^4 (\vec{q}^2)^2 \\ &\equiv \frac{1}{2} [\vec{p}^2 + m_0^2 \vec{q}^2] + \lambda_0 (\vec{q}^2)^2, \end{aligned} \quad (29)$$

which, according to (18), is exactly the expectation value we sought. Yes, conventional quantization requires that the canonical operators need to be realized by an irreducible representation, but that is no longer the case for enhanced quantization since the classical-quantum connection is fundamentally different in the two cases. For enhanced quantization, the classical-quantum connection is generically given by $H(\vec{p}, \vec{q}) \equiv \langle \vec{p}, \vec{q} | \mathcal{H} | \vec{p}, \vec{q} \rangle$. For the present model, $|\vec{p}, \vec{q}\rangle = |\vec{p}, \vec{q}; \zeta\rangle$, and the classical-quantum connection specifically means that

$$\begin{aligned} H(\vec{p}, \vec{q}) &\equiv \langle \vec{p}, \vec{q}; \zeta | \mathcal{H}_1(\vec{P}, \vec{R}; \vec{Q}, \vec{S}) | \vec{p}, \vec{q}; \zeta \rangle \\ &= \frac{1}{2}[\vec{p}^2 + m_0^2 \vec{q}^2] + \lambda_0 (\vec{q}^2)^2, \end{aligned} \quad (30)$$

with no contradiction whatsoever. Note that this feature of enhanced quantization is distinct from affine quantization, and leads to acceptable results for $N \leq \infty$. Significantly, when $N = \infty$, there are *no divergences*, unlike the quantization of most interacting models with an infinite number of degrees of freedom.

The present models have been studied previously [6, 7], and some spectral analysis of the interacting Hamiltonian has also been carried out in [8].

5 CONCLUSION

Conventional canonical quantization, with its direct association between classical variables and quantum variables is correct most of the time, but there are occasions when it leads to unnatural results. Also classical theory assumes that $\hbar = 0$ while quantum theory requires that $\hbar > 0$; however, in the real world $\hbar > 0$ and the classical theory must adapt to that fact, and this is exactly what enhanced quantization achieves. In addition, enhanced quantization offers untapped riches in the very concept of quantization as illustrated by the topic of Sec. III and the examples in Sec. IV. Further discussion of these new procedures as well as some additional examples may be found in [5, 9, 10, 11] for some simple models, including an example with spin variables, and [12, 13] for several field-theoretic models that, when conventionally quantized, violate the natural rule that the classical limit should result in the original classical theory, but, instead, are successfully quantized using enhanced quantization procedures.

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